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s-Degenerate curves in Lorentzian space forms☆

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Abstract

In this paper we introduce s-degenerate curves in Lorentzian space forms as those ones whose derivative of order s is a null vector provided that $s > 1$ and all derivatives of order less than s are space-like (see the exact definition in Section 2). In this sense classical null curves are 1-degenerate curves. We obtain a reference along an s-degenerate curve in an n-dimensional Lorentzian space with the minimum number of curvatures. That reference generalizes the reference of Bonnor for null curves in Minkowski space–time and it will be called the Cartan frame of the curve. The associated curvature functions are called the Cartan curvatures of the curve. We characterize the s-degenerate helices (i.e. s-degenerate curves with constant Cartan curvatures) in n-dimensional Lorentzian space forms and we obtain a complete classification of them in dimension four.

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1. Introduction

The geometry of null hypersurfaces in space–times has played an important role in the development of general relativity, as well as in mathematics and physics of gravitation. It is necessary, e.g. to understand the causal structure of space–times, black holes, assymptotically flat systems and gravitational waves.

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A starting point to study null surfaces, or in general null hypersurfaces, consists of investigating the curves that live in those hypersurfaces. In this sense, the null curves in Lorentzian space forms has been studied by several authors (see, e.g. [2,3,9], and references therein).

However, in a null hypersurface there are many other curves distinct from the null ones. They are space-like curves with a null higher derivative, i.e. s-degenerate curves (see Section 2 for details). In this paper we study s-degenerate curves in Lorentzian space forms $\mathbb{M}_1^n$ and obtain existence, uniqueness and congruence theorems for that kind of curves. Notice that they must be space-like curves.

Time-like and light-like trajectories are the natural ones in space–time geometries, but some recent experiments point out the existence of superluminal particles (space-like trajectories) without any breakdown of the principle of relativity; theoretical developments exist suggesting that neutrinos might be instances of "tachyons" as their square mass appears to be negative. A model has been recently presented to fit the cosmic ray spectrum at $E \approx 1-4\,\text{PeV}$ [6–8], using the hypothesis that the electron neutrino is a tachyon. This model yields a value for $m^2(\nu_e) \approx -3\,\text{eV}^2$, which is consistent with the results from recent measurements in tritium beta decay experiments [4,11,14]. Moreover, the muon neutrino also exhibits a negative mass-squared [1]. However, as it is pointed out in [5], at present time we have not a satisfactory quantum theory for tachyonic fermions, so more theoretical work would be needed to determine a physically acceptable theory.

In [12] the author considers a model of a D-dimensional massless particle described by a Lagrangian proportional to the Nth extrinsic curvature of the world-line. He presents the Hamiltonian formulation of the system and shows that its trajectories are space-like curves.

Therefore, it is required to construct a complete (at least local) theory of space-like trajectories for neutrinos. Here, we are intended to provide a suitable mathematical machinery to support the recent advances in theoretical physics.

In this paper we prove the following theorems.

Theorem 1.1. *Let $k_1, \ldots, k_m : [-\delta, \delta] \to \mathbb{R}$ be differentiable functions with $k_i > 0$ for $i \neq s, m$. Let p be a point in $\mathbb{M}_1^n$, $n = m + 2$, and let $\{W_1^0, \ldots, W_{s-1}^0, L^0, W_s^0, N^0, W_{s+1}^0, \ldots, W_m^0\}$ be a positively oriented pseudo-orthonormal basis of $T_p\mathbb{M}_1^n(c)$. Then there exists a unique s-degenerate Cartan curve γ in $\mathbb{M}_1^n(c)$, with $\gamma(0) = p$, whose Cartan reference satisfies*

$$L(0) = L^0, \qquad N(0) = N^0, \qquad W_i(0) = W_i^0, \quad i \in \{1, \ldots, m\}.$$

Theorem 1.2. *If two s-degenerate Cartan curves C and $\bar{C}$ in $\mathbb{M}_1^n(c)$ have Cartan curvatures $\{k_1, \ldots, k_m\}$, where $k_i : [-\delta, \delta] \to \mathbb{R}$ are differentiable functions, then there exists a Lorentzian transformation of $\mathbb{M}_1^n(c)$ which maps bijectively C into $\bar{C}$.*

In Section 5 we characterize the 2-degenerate helices (i.e. 2-degenerate curves with constant Cartan curvatures) in four-dimensional Lorentzian space forms and we obtain a complete classification of them.

2. Frenet frames for s-degenerate curves

The goal of this section is to find Frenet frames for s-degenerate curves in Lorentzian space forms. Before to do that, we need a technical result.

Let E be a real vector space with a symmetric bilinear mapping $g : E \times E \to \mathbb{R}$. We say that g is *degenerate* on E if there exists a vector $\xi \neq 0$ in E such that

$$g(\xi, v) = 0 \quad \text{for all} \ \ v \in E,$$

otherwise, g is said to be *non-degenerate*. The *radical* (also called the *null space*) of E, with respect to g, is the subspace $\mathrm{Rad}(E)$ of E defined by

$$\mathrm{Rad}(E) = \{\xi \in E | g(\xi, v) = 0, \quad v \in E\}.$$

The dimension of $\mathrm{Rad}(E)$ is called the *nullity degree* of g (or E) and is denoted by r_E.

If F is a subspace of E, then we can consider g_F the symmetric bilinear mapping on $F \times F$ obtained by restricting g and define r_F as the nullity degree of F (or g_F). For simplicity, we will use $\langle , \rangle$ instead of g or g_F.

A vector v is said to be *time-like, light-like* or *space-like* provided that $g(v, v) < 0$, $g(v, v) = 0$ (and $v \neq 0$), or $g(v, v) > 0$, respectively. The vector $v = 0$ is assumed to be space-like. A *unit vector* is a vector u such that $g(u, u) = \pm 1$.

Two vectors u and v are said to be orthogonal, written $u \perp v$, if $g(u, v) = 0$. Similarly, two subsets U and V of E are said to be *orthogonal* if $u \perp v$ for any $u \in U$ and $v \in V$. Given two orthogonal subspaces F_1 and F_2 in E with $F_1 \cap F_2 = \{0\}$, the orthogonal direct sum of F_1 and F_2 will be denoted by $F_1 \perp F_2$.

Lemma 2.1. *Let $(E, \langle , \rangle)$ be a bilinear space and let F be a hyperplane of E. Let $r_F = \dim \mathrm{Rad}(F)$ and $r_E = \dim \mathrm{Rad}(E)$. Then the following statements hold.*

(i) *If $r_F = 0$ and $r_E = 1$, then there exists a null vector L such that*

$$E = F \perp \mathrm{span}\{L\}.$$

(ii) *If $r_F = r_E \in \{0, 1\}$, then there exists a non-null unit vector V such that*

$$E = F \perp \mathrm{span}\{V\}.$$

Moreover, if $\mathrm{Rad}(E) = \{0\}$ then V is unique, up to the sign.

(iii) *If $r_F = 1$ and $r_E = 0$, and $F = F_1 \perp L$, where $L \in \mathrm{Rad}(F)$ and F_1 is non-degenerate, then there exists a unique null vector N such that $\langle L, N\rangle = \varepsilon$, $\varepsilon = \pm 1$, and*

$$E = (\mathrm{span}\{L\} \oplus \mathrm{span}\{N\}) \perp F_1.$$

Proof. We only need to make some algebraic computations.

(i) Since F is non-degenerate, then $E = F \perp F^{\perp}$, where $F^{\perp} = \mathrm{span}\{L\}$ for a certain vector L. The inclusion $\mathrm{Rad}(E) \subset F^{\perp}$ implies $\mathrm{Rad}(E) = F^{\perp}$ and so L is a null vector.

(ii) We may assume that $r_F = r_E = 1$. By considering $F = F_1 \perp \mathrm{span}\{L\}$, where F_1 is non-degenerate and L is null, then $E = F_1 \perp F_1^{\perp}$. Since $\dim F_1^{\perp} = 2$, then $F_1^{\perp} = \mathrm{span}\{L\} \oplus \mathrm{span}\{V\}$, where $\mathrm{Rad}(E) = \mathrm{span}\{L\}$ and V is a non-null vector in $F^{\perp}$, so that the required splitting is fulfilled.

(iii) By a similar reasoning we may assume that $F = F_1 \perp \mathrm{span}\{L\}$, where $F_1^{\perp} = \mathrm{span}\{L\} \oplus \mathrm{span}\{V\}$. Since $\mathrm{Rad}(E) = \{0\}$ then $\langle L, V\rangle \neq 0$. Let N be the vector defined by

$$N = \frac{\varepsilon}{\langle L, V\rangle}\left(V - \frac{\langle V, V\rangle}{2\langle L, V\rangle} L\right).$$

It is easy to see that N is the only vector satisfying $\langle N, N\rangle = 0$, $\langle L, N\rangle = \varepsilon$ and $N \in F_1^{\perp}$, and the splitting follows. □

Let (M_1^n, ∇) be an oriented Lorentzian manifold and let $\gamma : I \to M_1^n$ be a differentiable curve in M_1^n. For any vector field V along γ, let V' be the covariant derivative of V along γ. Write $E_i(t) = \mathrm{span}\{\gamma'(t), \gamma''(t), \dots, \gamma^{(i)}(t)\}$, where $t \in I$ and $i = 1, 2, \dots, n$. Let d be the number defined by $d = \max\{i : \dim E_i(t) = i \ \text{ for all } t\}$.

Definition 2.2. With the above notations, the curve $\gamma : I \to M_1^n$ is said to be an *s-degenerate* (*or s-light-like*) *curve* if for all $1 \leq i \leq d$, $\dim \mathrm{Rad}(E_i(t))$ is constant for all t, and there exists s, $0 < s \leq d$, such that $\mathrm{Rad}(E_s) \neq \{0\}$ and $\mathrm{Rad}(E_j) = \{0\}$ for all $j < s$.

Remark 2.3. Note that 1-degenerate curves are precisely the null (or light-like) curves (see, for instance [2,3,9], and references therein). In this paper we will focus on s-degenerate curves ($s > 1$), in Lorentzian spaces. Notice that they must be space-like curves.

To find the Frenet frames, we will distinguish four cases separately:

1) $d = n$ and $s \leq d$;
2) $d < n$ and $s = d$;
3) $d < n$ and $s = d - 1$;
4) $d < n$ and $s < d - 1$.

Case 2.4 ($d = n$ and $s \leq d$). First of all, write $\gamma' = \bar{k}_1 W_1$, where W_1 is a unit space-like vector such that $\bar{k}_1 > 0$. Then $E_2 = \mathrm{span}\{W_1\} \oplus \mathrm{span}\{\gamma''\}$, so that from Lemma 2.1 there exists a unit space-like vector W_2 such that $E_2 = \mathrm{span}\{W_1\} \perp \mathrm{span}\{W_2\}$. Furthermore, W_2 is unique by choosing it in such a way that $\{\gamma', \gamma''\}$ and $\{W_1, W_2\}$ have the same orientation. By proceeding iteratively, using Lemma 2.1, we obtain a set $\{W_1, \dots, W_{s-1}\}$ of orthonormal space-like sections along γ such that $\{\gamma', \dots, \gamma^{(i)}\}$ and $\{W_1, \dots, W_i\}$ have the same orientation for all i, $1 \leq i \leq s - 1$. Now we have $E_s = E_{s-1} \oplus \mathrm{span}\{\gamma^{(s)}\}$ and $\dim \mathrm{Rad}(E_s) = 1$. By using again Lemma 2.1 we can find a (not unique) null vector field L such that $E_s = E_{s-1} \oplus \mathrm{span}\{L\}$. As $s \neq n$, because E_n is non-degenerate, then $E_{s+1} = E_s \oplus \mathrm{span}\{\gamma^{(s+1)}\}$. Now we will prove that $\dim \mathrm{Rad}(E_{s+1}) = 1$. By assuming that $\dim \mathrm{Rad}(E_{s+1}) = 0$, then there exists a unique null vector field N satisfying $\langle W_i, N\rangle = \langle N, N\rangle = 0$, $\langle L, N\rangle = \varepsilon$, $\varepsilon = \pm 1$, and $E_{s+1} = \mathrm{span}\{W_1, \dots, W_{s-1}, L, N\}$. By taking derivatives we obtain the following equations:

$$\gamma' = \bar{k}_1 W_1, \qquad W_1' = \bar{k}_2 W_2, \qquad W_i' = -\bar{k}_i W_{i-1} + \bar{k}_{i+1} W_{i+1}, \quad 2 \leq i \leq s-2,$$
$$W_{s-1}' = -\bar{k}_{s-1} W_{s-2} + \varepsilon \bar{k}_s L, \qquad L' = \varepsilon \bar{k}_{s+1} L$$

for certain functions $\bar{k}_j$, $j = 1, \dots, s+1$. As $L \in \text{span}\{\gamma', \dots, \gamma^{(s)}\}$, we can write $L = \lambda_1 \gamma' + \cdots + \lambda_s \gamma^{(s)}$, with $\lambda_s \neq 0$, and therefore, $L' = (*) + \lambda_s \gamma^{(s+1)} = \varepsilon \bar{k}_{s+1} L \in \text{span}\{\gamma', \dots, \gamma^{(s)}\}$. We conclude that $\gamma^{(s+1)} \in \text{span}\{\gamma', \dots, \gamma^{(s)}\}$, which cannot hold.

Then $\dim \text{Rad}(E_{s+1}) = 1$, and using Lemma 2.1 once more there exists a (not unique) vector field W_s such that $\{\gamma', \dots, \gamma^{(s+1)}\}$ and $\{W_1, \dots, W_{s-1}, L, W_s\}$ have the same orientation. Since $n > s+1$, we claim that $\dim \text{Rad}(E_{s+2}) = 0$. Otherwise, there exists a unit space-like vector field W_{s+1} orthogonal to E_{s+1}. By differentiating we obtain

$$W_{s-1}' = -\bar{k}_{s-1} W_{s-2} + \varepsilon \bar{k}_s L, \qquad L' = \varepsilon \bar{k}_{s+1} L + \bar{k}_{s+2} W_s. \tag{1}$$

Since $\text{Rad}(E_{s+2}) = \text{span}\{L\}$ we get $\langle L, \gamma^{(s+1)} \rangle = \langle L, \gamma^{(s+2)} \rangle = 0$, so that $\langle L', \gamma^{(s+1)} \rangle = 0$. From here and (1) we find that $\langle W_s, \gamma^{(s+1)} \rangle = 0$ (i.e. W_s lies in $\text{Rad}(E_{s+1})$), which is a contradiction. Hence $\dim \text{Rad}(E_{s+2}) = 0$ and there exists a unique N satisfying $\langle N, L \rangle = \varepsilon$ and $\langle N, W_i \rangle = 0$. We choose ε in such a way that $\{\gamma', \dots, \gamma^{(s+2)}\}$ and $\{W_1, \dots, W_{s-1}, L, W_s, N\}$ have the same orientation. If $s+2 = n$, the process concludes; otherwise, $\dim \text{Rad}(E_i) = 0$ for $i > s+2$ and we can obtain orthonormal space-like sections $\{W_{s+1}, \dots, W_m\}$, $m = n-2$, with the same orientation rule. The vector field W_m is chosen in order that $\{W_1, \dots, W_{s-1}, L, W_s, N, W_{s+1}, \dots, W_m\}$ is positively oriented. Regarding this reference, we have the following equations:

$$\gamma' = \bar{k}_1 W_1, \qquad W_1' = \bar{k}_2 W_2, \qquad W_i' = -\bar{k}_i W_{i-1} + \bar{k}_{i+1} W_{i+1}, \quad 2 \leq i \leq s-2,$$
$$W_{s-1}' = -\bar{k}_{s-1} W_{s-2} + \varepsilon \bar{k}_s L, \qquad L' = \varepsilon \bar{k}_{s+1} L + \bar{k}_{s+2} W_s,$$
$$W_s' = \varepsilon \bar{k}_{s+3} L - \varepsilon \bar{k}_{s+2} N, \qquad N' = -\bar{k}_s W_{s-1} - \varepsilon \bar{k}_{s+1} N - \bar{k}_{s+3} W_s + \bar{k}_{s+4} W_{s+1},$$
$$W_{s+1}' = -\varepsilon \bar{k}_{s+4} L + \bar{k}_{s+5} W_{s+2}, \qquad W_j' = -\bar{k}_{j+3} W_{j-1} + \bar{k}_{j+4} W_{j+1},$$
$$s+2 \leq j \leq m-1, \qquad W_m' = -\bar{k}_{m+3} W_{m-1}$$

for certain functions $\{\bar{k}_1, \dots, \bar{k}_{m+3}\}$. The set $\mathcal{F} = \{W_1, \dots, W_{s-1}, L, W_s, N, W_{s+1}, \dots, W_m\}$ is said to be a *Frenet reference* along γ. The functions $\{\bar{k}_1, \dots, \bar{k}_{m+3}\}$ are called the *curvature functions* of γ with respect to $\mathcal{F}$. The above equations are called the *Frenet equations* of γ with respect to $\mathcal{F}$.

Case 2.5 ($d < n$ and $s = d$). A similar reasoning as in Case 2.4 shows that there exists a set $\{W_1, \dots, W_{s-1}, L\}$ such that L is a null vector, $\{W_1, \dots, W_{s-1}\}$ is an orthonormal set of space-like vectors and $E_d = \text{span}\{W_1, \dots, W_{s-1}, L\}$. Then we can obtain the following equations:

$$\gamma' = \bar{k}_1 W_1, \qquad W_1' = \bar{k}_2 W_2, \qquad W_i' = -\bar{k}_i W_{i-1} + \bar{k}_{i+1} W_{i+1}, \quad 2 \leq i \leq s-2,$$
$$W_{s-1}' = -\bar{k}_{s-1} W_{s-2} + \varepsilon \bar{k}_s L, \qquad L' = \varepsilon \bar{k}_{s+1} L$$

for certain functions $\{\bar{k}_1, \dots, \bar{k}_{s+1}\}$. If M_1^n is a Lorentzian space form, then γ lies in a d-dimensional totally geodesic light-like submanifold. This can be proved by adapting the proofs of Theorems 5 and 9 of Chapter 7 in [13]. This case has been treated in Section 2 of [10].

Case 2.6 ($d < n$ and $s = d - 1$). As above again, we obtain $E_d = \text{span}\{W_1, \dots, W_{s-1}, L, W_s\}$ and equations

$$W'_{s-1} = -\bar{k}_{s-1}W_{s-2} + \varepsilon\bar{k}_s L, \qquad L' = \varepsilon\bar{k}_{s+1}L + \bar{k}_{s+2}W_s, \qquad W'_s = \varepsilon\bar{k}_{s+3}L.$$

Since W_s lies in $E_s^\perp$, we have $\langle W_s, \gamma^{(s)}\rangle = 0$. By differentiating here we deduce that $\langle W_s, \gamma^{(s+1)}\rangle = 0$, which is a contradiction.

Case 2.7 ($d < n$ and $s < d-1$). Now we have $E_d = \{W_1, \dots, W_{s-1}, L, W_s, N, W_{s+1}, \dots, W_{d-2}\}$. Working as in case of non-degenerate curves (see, e.g. [13, Vol. IV]), if M_1^n is a Lorentzian space form we deduce that γ lies in a d-dimensional non-degenerate totally geodesic submanifold of M_1^n. So this case reduces to Case 2.4.

Remark 2.8. Before going any further, we note that the type s does not depend on the parameter of the curve. To see that let $\bar{t}$ be another parameter and write $\gamma(t) = \beta(\bar{t}(t))$. By differentiating with respect to t we get $\gamma^{(i)}(t) = \sum_{j=1}^{i} x_{ij}(t)\beta^j(\bar{t})$, i.e. $E_i = \text{span}\{\gamma'(t), \dots, \gamma^{(i)}(t)\} = \text{span}\{\beta'(\bar{t}), \dots, \beta^{(i)}(\bar{t})\}$, which shows the claim.

On the other hand, let $\Phi : M_1^n \to M_1^n$ be an isometry and $\bar{\gamma}(t) = (\Phi \circ \gamma)(t)$. Then for all vector field V along γ we have

$$\frac{\bar{D}}{\mathrm{d}t}(\mathrm{d}\Phi_{\gamma(t)}(V(t))) = \mathrm{d}\Phi_{\gamma(t)}\left(\frac{D}{\mathrm{d}t}V(t)\right),$$

where D_t and $\bar{D}_t$ stand for the covariant derivatives along γ and $\bar{\gamma}$, respectively.

Hence $\langle\gamma^{(i)}(t), \gamma^{(j)}(t)\rangle = \langle\bar{\gamma}^{(i)}(t), \bar{\gamma}^{(j)}(t)\rangle$ showing that this kind of curves are invariant under Lorentzian transformations, in the sense that the type s does not change under a Lorentzian transformation.

3. The Cartan reference of an s-degenerate curve

The goal of this section is to find a Frenet frame with the minimal number of curvatures and such that they are invariant under Lorentzian transformations. We will restrict ourselves to Case 2.4. Without loss of generality, let us assume that γ is arc-length parametrized, so that $W_1 = \gamma'$ and $\bar{k}_1 = 1$. By taking $\bar{k}_s = \varepsilon$, Lemma 2.1 leads to a uniquely determined set $\{W_1, \dots, W_{s-1}, L\}$. Therefore, we only need to find W_s.

Suppose that W_s and W_s^* are two distinct vector fields generating two distinct Frenet frames, i.e.:

$$\{W_1, \dots, W_{s-1}, L, W_s, N, W_{s+1}, \dots, W_m\} \to \{\bar{k}_1 = 1, \bar{k}_2, \dots, \bar{k}_s = 1, \bar{k}_{s+1}, \dots, \bar{k}_{m+3}\},$$
$$\{W_1, \dots, W_{s-1}, L, W_s^*, N^*, W_{s+1}^*, \dots, W_m^*\} \to \{\bar{k}_1 = 1, \bar{k}_2, \dots, \bar{k}_s = 1, \bar{k}_{s+1}^*, \dots, \bar{k}_{m+3}^*\}.$$

A straightforward computation shows that

$$W_s^* = fL + W_s, \qquad N^* = -\tfrac{1}{2}f^2L + N - fW_s, \qquad \bar{k}_{s+1}^* = \bar{k}_{s+1} - f\bar{k}_{s+2}, \tag{2}$$

where $f : I \to \mathbb{R}$ is a differentiable function. We can choose f in such a way that $\bar{k}_{s+1}^* = 0$. Then by reordering the curvature functions we have the following equations:

$$\begin{aligned}
&\gamma' = W_1, \qquad W_1' = k_1W_2, \qquad W_i' = -k_{i-1}W_{i-1} + k_iW_{i+1}, \quad 2 \le i \le s-2,\\
&W_{s-1}' = -k_{s-2}W_{s-2} + L, \qquad L' = k_{s-1}W_s, \qquad W_s' = \varepsilon k_sL - \varepsilon k_{s-1}N,\\
&N' = -\varepsilon W_{s-1} - k_sW_s + k_{s+1}W_{s+1}, \qquad W_{s+1}' = -\varepsilon k_{s+1}L + k_{s+2}W_{s+2},\\
&W_j' = -k_jW_{j-1} + k_{j+1}W_{j+1}, \quad s+2 \le j \le m-1, \qquad W_m' = -k_mW_{m-1}
\end{aligned} \tag{3}$$

for certain functions $\{k_1, \dots, k_m\}$. Bearing in mind (2) we can easily deduce the following result.

Theorem 3.1. *Let $\gamma : I \to M_1^n$, $n = m+2$, be an s-degenerate unit curve, $s > 1$, and suppose that $\{\gamma'(t), \gamma''(t), \dots, \gamma^{(n)}(t)\}$ spans $T_{\gamma(t)}M_1^n$ for all t. Then there exists a unique Frenet frame satisfying* Eq. (3).

Definition 3.2. An s-degenerate curve, $s > 1$, satisfying the above conditions is said to be an *s-degenerate Cartan curve*. The reference and curvature functions given by (3) will be called the *Cartan reference* and *Cartan curvatures* of γ, respectively.

Observe that when $m > s$ then $\varepsilon = -1$ and $k_i > 0$ for $i \neq s$, and $k_m > 0$ or ($k_m < 0$, resp.) according to $\{\gamma', \gamma'', \dots, \gamma^{(n)}\}$ is positively or negatively oriented, respectively. However, when $m = s$ then $\varepsilon = -1$ or $\varepsilon = 1$ according to $\{\gamma', \gamma'', \dots, \gamma^{(n)}\}$ is positively or negatively oriented, respectively, and $k_i > 0$ for $i \neq s$.

Definition 3.3. An *s-degenerate helix* in M_1^n is an s-degenerate Cartan curve having constant Cartan curvatures.

4. s-Degenerate curves in Lorentzian space forms

Let $\gamma : I \to \mathbb{M}_1^n(c)$ be an s-degenerate Cartan curve, $\mathbb{M}_1^n(c)$ standing for $\mathbb{R}_1^n$, $\mathbb{S}_1^n \circ \mathbb{H}_1^n$, according to $c = 0$, $c = 1$ or $c = -1$, respectively. Let D_t denote the covariant derivative in $\mathbb{M}_1^n(c)$ along γ. Then for any vector field V along γ we have $D_tV = V' + c\langle V, \gamma'\rangle\gamma$, where $\langle , \rangle$ denotes the standard metric in $\mathbb{R}_1^n$, $\mathbb{R}_1^{n+1}$ or $\mathbb{R}_2^{n+1}$. If $\{W_1, \dots, W_{s-1}, L, W_s, N, W_{s+1}, \dots, W_m\}$ is the Cartan reference, then Eq. (3) can be written as follows:

$$\begin{aligned}
&\gamma' = W_1, \qquad W_1' = k_1W_2 - c\gamma, \qquad W_i' = -k_{i-1}W_{i-1} + k_iW_{i+1}, \quad 2 \le i \le s-2,\\
&W_{s-1}' = -k_{s-2}W_{s-2} + L, \qquad L' = k_{s-1}W_s, \qquad W_s' = \varepsilon k_sL - \varepsilon k_{s-1}N,\\
&N' = -\varepsilon W_{s-1} - k_sW_s + k_{s+1}W_{s+1}, \qquad W_{s+1}' = -\varepsilon k_{s+1}L + k_{s+2}W_{s+2},\\
&W_j' = -k_jW_{j-1} + k_{j+1}W_{j+1}, \quad s+2 \le j \le m-1, \qquad W_m' = -k_mW_{m-1}.
\end{aligned} \tag{4}$$

Now we state the following question: Let $\{W_1, \dots, W_{s-1}, L, W_s, N, W_{s+1}, \dots, W_m\}$ be a reference satisfying (3) for certain functions k_j. Is there an s-degenerate Cartan curve γ having $\{W_1, \dots, W_{s-1}, L, W_s, N, W_{s+1}, \dots, W_m\}$ as Cartan reference and k_j as Cartan curvatures?

The answer is affirmative, as we will show in this section. But before to do that, we are going to state and prove an algebraic result.

Definition 4.1. A basis $\mathcal{B} = \{L_1, N_1, \dots, L_r, N_r, W_1, \dots, W_m\}$ of $\mathbb{R}^n_q$, with $2r \le 2q \le n$ and $m = n - 2r$, is said to be *pseudo-orthonormal* if it satisfies the following equations:

$$\langle L_i, L_j\rangle = \langle N_i, N_j\rangle = 0, \quad \langle L_i, N_i\rangle = \varepsilon_i, \ \ \langle L_i, N_j\rangle = 0, \ \ i \ne j,$$
$$\langle L_i, W_\alpha\rangle = \langle N_i, W_\alpha\rangle = 0, \quad \langle W_\alpha, W_\beta\rangle = \varepsilon_\alpha \delta_{\alpha\beta},$$

where $i, j \in \{1, \dots, r\}$, $\alpha, \beta \in \{1, \dots, m\}$, $\varepsilon_\alpha = -1$ if $1 \le \alpha \le q - r$ and $\varepsilon_\alpha = 1$ if $q - r + 1 \le \alpha \le m$.

Lemma 4.2. *Let $\mathcal{B} = \{L_1, N_1, \dots, L_r, N_r, W_1, \dots, W_m\}$ be a basis of $\mathbb{R}^n_q$, with $2r \le 2q \le n$ and $m = n - 2r$. Consider $\mathcal{B}' = \{V_1, \dots, V_q, V_{q+1}, \dots, V_n\}$ where*

$$V_i = \begin{cases} \dfrac{1}{\sqrt{2}}(L_i - \varepsilon_i N_i) & i = 1, \dots, r, \\ W_{i-r} & i = r+1, \dots, q, \\ \dfrac{1}{\sqrt{2}}(L_{i-q} + \varepsilon_{i-q} N_{i-q}) & i = q+1, \dots, q+r, \\ W_{i-2r} & i = q+r+1, \dots, n. \end{cases} \tag{5}$$

The following conditions are equivalent:

(i) *$\mathcal{B}$ is a pseudo-orthonormal basis.*
(ii) *$\mathcal{B}'$ is an orthonormal basis.*
(iii) *$\mathcal{B}'$ satisfies*

$$-\sum_{\alpha=1}^{q} V_{\alpha i} V_{\alpha j} + \sum_{\beta=q+1}^{n} V_{\beta i} V_{\beta j} = \eta_{ij}.$$

(iv) *$\mathcal{B}$ satisfies*

$$\sum_{\alpha=1}^{r} \varepsilon_\alpha (L_{\alpha i} N_{\alpha j} + L_{\alpha j} N_{\alpha i}) - \sum_{\beta=1}^{q-r} W_{\beta i} W_{\beta j} + \sum_{\theta=q-r+1}^{m} W_{\theta i} W_{\theta j} = \eta_{ij}.$$

Here $V_{\rho k}$, $L_{\rho k}$, $N_{\rho k}$ and $W_{\rho k}$ stand for the components of vectors V_ρ, L_ρ, N_ρ and W_ρ, respectively, and (η_{ij}) the matrix of the canonical metric in the standard coordinates.

Proof. (i) $\Leftrightarrow$ (ii) It is obvious.

(ii) ⇔ (iii) Consider the matrices $V = (V_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ in $\mathcal{M}_{n\times n}(\mathbb{R})$ given by

$$b_{ij} = \langle V_i, V_j\rangle, \qquad c_{ij} = -\sum_{\alpha=1}^{q} V_{\alpha i}V_{\alpha j} + \sum_{\beta=q+1}^{n} V_{\beta i}V_{\beta j}.$$

Put

$$V = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \qquad C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},$$

where A_1, B_1 and C_1 are matrices in $\mathcal{M}_{q\times q}(\mathbb{R})$. Consider the complex matrix

$$A = \begin{pmatrix} A_1 & \mathrm{i}A_2 \\ -\mathrm{i}A_3 & A_4 \end{pmatrix} \in \mathcal{M}_{n\times n}(\mathbb{C}).$$

Then a straightforward computation shows that

$$AA^{\mathrm{T}} = \begin{pmatrix} -B_1 & \mathrm{i}B_2 \\ \mathrm{i}B_3 & B_4 \end{pmatrix}, \qquad A^{\mathrm{T}}A = \begin{pmatrix} -C_1 & -\mathrm{i}C_2 \\ -\mathrm{i}C_3 & C_4 \end{pmatrix}.$$

Then $\mathcal{B}'$ is orthonormal if and only if $C_1 = -I$, $C_4 = I$ and $C_2 = C_3 = 0$.

(iii) ⇔ (iv) From (5) we have

$$L_\alpha = \frac{1}{\sqrt{2}}(V_{\alpha+q} + V_\alpha), \qquad N_\alpha = \frac{\varepsilon_\alpha}{\sqrt{2}}(V_{\alpha+q} - V_\alpha), \qquad \alpha \in \{1,\ldots,r\}$$

and therefore

$$\varepsilon_\alpha(L_{\alpha i}N_{\alpha j} + N_{\alpha i}L_{\alpha j}) = -V_{\alpha i}V_{\alpha j} + V_{(\alpha+q)i}V_{(\alpha+q)j},$$
$$\alpha \in \{1,\ldots,r\}, \quad i,j \in \{1,\ldots,n\},$$

which finishes the proof. □

Theorem 4.3. *Let $k_1,\ldots,k_m : [-\delta,\delta] \to \mathbb{R}$ be differentiable functions with $k_i > 0$ for $i \neq s, m$. Let p be a point in $\mathbb{M}_1^n$, $n = m+2$, and let $\{W_1^0,\ldots,W_{s-1}^0, L^0, W_s^0, N^0, W_{s+1}^0,\ldots,W_m^0\}$ be a positively oriented pseudo-orthonormal basis of $T_p\mathbb{M}_1^n(c)$. Then there exists a unique s-degenerate Cartan curve γ in $\mathbb{M}_1^n(c)$, with $\gamma(0) = p$, whose Cartan reference satisfies*

$$L(0) = L^0, \qquad N(0) = N^0, \qquad W_i(0) = W_i^0, \quad i \in \{1,\ldots,m\}.$$

Proof. By the general theory of differential equations we know that there exists a unique solution $\{W_1,\ldots,W_{s-1}, L, W_s, N, W_{s+1},\ldots,W_m\}$ of (4), defined on the interval $[-\delta,\delta]$, and satisfying the initial conditions of the theorem. Taking into account (4), a straightforward computation leads to

$$\frac{\mathrm{d}t}{\mathrm{d}t}\left(\varepsilon(L_i(t)N_j(t) + L_j(t)N_i(t)) + c\gamma_i(t)\gamma_j(t) + \sum_{\beta=1}^{m} W_{\beta i}(t)W_{\beta j}(t)\right) = 0.$$

Now, since $\{W_1, \ldots, W_{s-1}, L, W_s, N, W_{s+1}, \ldots, W_m\}$ is pseudo-orthonormal at $t = 0$, Lemma 4.2 (with $r = 1$) yields

$$\varepsilon(L_i(t)N_j(t) + L_j(t)N_i(t)) + c\gamma_i(t)\gamma_j(t) + \sum_{\beta=1}^{m} W_{\beta i}(t)W_{\beta j}(t) = \nu_{ij} \quad \forall t \in [\delta, \delta].$$

By using again Lemma 4.2, we deduce that, for all t, $\{L, N, W_1, \ldots, W_m, \gamma\}$ is pseudo-orthonormal if $c = \pm 1$, and $\{L, N, W_1, \ldots, W_m\}$ is pseudo-orthonormal if $c = 0$. This concludes the proof. □

Theorem 4.4 (Congruence theorem). *If two s-degenerate Cartan curves C and $\bar{C}$ in $\mathbb{M}_1^n(c)$ have Cartan curvatures $\{k_1, \ldots, k_m\}$, where $k_i : [-\delta, \delta] \to \mathbb{R}$ are differentiable functions, then there exists a Lorentzian transformation of $\mathbb{M}_1^n(c)$ which maps bijectively C into $\bar{C}$.*

5. s-Degenerate helices in $\mathbb{M}_1^4(c)$

This section is devoted to the classification of 2-degenerate Cartan helices in Lorentzian space forms $\mathbb{M}_1^4(c)$. Now, the Cartan equations can be written as follows:

$$\gamma' = W_1, \qquad W_1' = L - c\gamma, \qquad L' = k_1 W_2, \qquad W_2' = \varepsilon k_2 L - \varepsilon k_1 N,$$
$$N' = -\varepsilon W_1 - k_2 W_2. \tag{6}$$

If we assume that k_1 and k_2 are constant, then γ satisfies the following differential equation:

$$\gamma^{(5)} - (2\varepsilon k_1 k_2 - c)\gamma^{(3)} - (k_1^2 + 2\varepsilon c k_1 k_2)\gamma' = 0.$$

Without loss of generality, we can assume that γ is positively oriented, i.e. $\varepsilon = -1$.

In what follows, we will present examples of 2-degenerate Cartan helices in $\mathbb{M}_1^4(c)$ and show the corresponding characterization theorems.

5.1. *Helices in $\mathbb{R}_1^4$*

Example 5.1. Let $\gamma_{\omega,\sigma}$ be the curve in $\mathbb{R}_1^4$ defined by

$$\gamma_{\omega,\sigma}(t) = \frac{1}{\sqrt{\omega^2 + \sigma^2}}\left(\frac{\sigma}{\omega}\cosh \omega t, \frac{\sigma}{\omega}\sinh \omega t, \frac{\omega}{\sigma}\sin \sigma t, \frac{\omega}{\sigma}\cos \sigma t\right)$$

with $\omega\sigma > 0$. Then $\gamma_{\omega,\sigma}$ is a helix with curvatures

$$k_1 = \omega\sigma, \qquad k_2 = \frac{\sigma^2 - \omega^2}{2\omega\sigma}.$$

Theorem 5.2 (Clasification theorem of 2-degenerate helices in $\mathbb{R}_1^4$). *Let γ be an s-degenerate Cartan curve fully immersed in $\mathbb{R}_1^4$. Then γ is a helix if and only if it is congruent to a helix of Example 5.1.*

Proof. Let $k_1 > 0$ and k_2 be the constant curvatures of γ. By Theorem 1.2 it suffices to find a helix of the family given in Example 5.1 with these curvatures. Take constants ω and σ such that

$$\omega^2 = k_1\left(-k_2 + \sqrt{1+k_2^2}\right), \qquad \sigma^2 = k_1\left(k_2 + \sqrt{1+k_2^2}\right)$$

with $\omega\sigma > 0$. The proof concludes since the curvatures of $\gamma_{\omega,\sigma}$ are k_1 and k_2. □

5.2. *Helices in* $\mathbb{S}_1^4$

Example 5.3 (Helices of type 1). Let $0 < \sigma^2 < 1 < \omega^2$ and let $\gamma_{\omega,\sigma}$ be the curve in $\mathbb{S}_1^4$ defined by

$$\gamma_{\omega,\sigma}(t) = \left(\sqrt{\frac{(\omega^2-1)(1-\sigma^2)}{\omega^2\sigma^2}}, \frac{1}{\omega}\sqrt{\frac{1-\sigma^2}{\omega^2-\sigma^2}}\sin\omega t, \frac{1}{\omega}\sqrt{\frac{1-\sigma^2}{\omega^2-\sigma^2}}\cos\omega t, \frac{1}{\sigma}\sqrt{\frac{\omega^2-1}{\omega^2-\sigma^2}}\sin\sigma t, \frac{1}{\sigma}\sqrt{\frac{\omega^2-1}{\omega^2-\sigma^2}}\cos\sigma t\right).$$

Then $\gamma_{\omega,\sigma}$ is a helix with curvatures

$$k_1 = \sqrt{(\omega^2-1)(1-\sigma^2)}, \qquad k_2 = \frac{\omega^2+\sigma^2-1}{2\sqrt{(\omega^2-1)(1-\sigma^2)}}.$$

Example 5.4 (Helices of type 2). Let $\sigma^2 > 1$ and let $\gamma_{\omega,\sigma}$ be the curve in $\mathbb{S}_1^4$ defined by

$$\gamma_{\omega,\sigma}(t) = \left(\frac{1}{\omega}\sqrt{\frac{\sigma^2-1}{\omega^2+\sigma^2}}\cosh\omega t, \frac{1}{\omega}\sqrt{\frac{\sigma^2-1}{\omega^2+\sigma^2}}\sinh\omega t, \frac{1}{\sigma}\sqrt{\frac{\omega^2+1}{\omega^2+\sigma^2}}\sin\sigma t, \frac{1}{\sigma}\sqrt{\frac{\omega^2+1}{\omega^2+\sigma^2}}\cos\sigma t, \frac{1}{\omega\sigma}\sqrt{(\omega^2+1)(\sigma^2-1)}\right), \quad \omega \neq 0.$$

Then $\gamma_{\omega,\sigma}$ is a helix with curvatures

$$k_1 = \sqrt{(\sigma^2-1)(\omega^2+1)}, \qquad k_2 = \frac{\sigma^2-\omega^2-1}{2\sqrt{(\sigma^2-1)(\omega^2+1)}}.$$

Example 5.5 (Helices of type 3). Let $\sigma^2 > 1$ and let γ_σ be the curve in $\mathbb{S}_1^4$ defined by

$$\gamma_\sigma(t) = \left(\frac{1}{2}\frac{\sqrt{\sigma^4-1}}{\sigma^2-1}t^2, \sqrt{\frac{\sigma^2-1}{\sigma^2}}t, \frac{\sqrt{\sigma^4-1}}{\sigma^2} - \frac{\sqrt{\sigma^4-1}}{2(\sigma^2+1)}t^2, \frac{1}{\sigma^2}\sin\sigma t, \frac{1}{\sigma^2}\cos\sigma t\right).$$

Then γ_σ is a helix with curvatures

$$k_1 = \sqrt{\sigma^2 - 1}, \qquad k_2 = \tfrac{1}{2}(\sqrt{\sigma^2 - 1}).$$

Theorem 5.6 (Classification theorem of 2-degenerate helices in $\mathbb{S}^4_1$). *Let γ be an s-degenerate Cartan curve fully immersed in $\mathbb{S}^4_1$. Then γ is a helix if and only if it is congruent to one in the families described in* Examples 5.3–5.5.

Proof. Let $k_1 > 0$ and k_2 be the constant curvatures of γ. We have to find a helix of one of the above types with these curvatures.

Case 1. Assume that $k_2 > k_1/2$. Take the helix $\gamma_{\omega,\sigma}$ of type 1 determined by

$$\omega^2 = \tfrac{1}{2}((2k_1k_2 + 1) + \sqrt{(1 - 2k_1k_2)^2 + 4k_1^2}),$$
$$\sigma^2 = \tfrac{1}{2}((2k_1k_2 + 1) - \sqrt{(1 - 2k_1k_2)^2 + 4k_1^2}).$$

A straightforward computation shows that $0 < \sigma^2 < 1 < \omega^2$ and the curvatures of $\gamma_{\omega,\sigma}$ are k_1 and k_2.

Case 2. Assume that $k_2 < k_1/2$. Take the helix $\gamma_{\omega,\sigma}$ of type 2 determined by

$$\omega^2 = \tfrac{1}{2}(-(2k_1k_2 + 1) + \sqrt{(1 - 2k_1k_2)^2 + 4k_1^2}),$$
$$\sigma^2 = \tfrac{1}{2}((2k_1k_2 + 1) + \sqrt{(1 - 2k_1k_2)^2 + 4k_1^2}).$$

It is easy to show that $\sigma^2 > 1$ and the curvatures of $\gamma_{\omega,\sigma}$ are k_1 and k_2.

Case 3. Assume that $k_2 = k_1/2$. Take the helix γ_σ of type 3 determined by $\sigma^2 = 1 + k_1^2$. It is easy to see that $\sigma^2 > 1$ and the curvatures of γ_σ are k_1 and k_2.

The result follows from Theorem 1.2. □

5.3. *Helices en $\mathbb{H}^4_1$*

Example 5.7 (Helices of type 1). Let $0 < \sigma^2 < 1 < \omega^2$ and let $\gamma_{\omega,\sigma}$ be the curve in $\mathbb{H}^4_1$ defined by

$$\gamma_{\omega,\sigma}(t) = \left(\frac{1}{\omega}\sqrt{\frac{1-\sigma^2}{\omega^2-\sigma^2}}\cosh \omega t, \frac{1}{\sigma}\sqrt{\frac{\omega^2-1}{\omega^2-\sigma^2}}\cosh \sigma t, \frac{1}{\omega}\sqrt{\frac{1-\sigma^2}{\omega^2-\sigma^2}}\sinh \omega t, \right.$$
$$\left. \frac{1}{\sigma}\sqrt{\frac{\omega^2-1}{\omega^2-\sigma^2}}\sinh \sigma t, -\frac{1}{\omega\sigma}\sqrt{(\omega^2-1)(1-\sigma^2)}\right).$$

Then $\gamma_{\omega,\sigma}$ is a helix with curvatures

$$k_1 = \sqrt{(\omega^2 - 1)(1 - \sigma^2)}, \qquad k_2 = -\frac{1}{2}\frac{\omega^2 + \sigma^2 - 1}{\sqrt{(\omega^2 - 1)(1 - \sigma^2)}}.$$

Example 5.8 (Helices of type 2). Let $\omega^2 > 1$ and let $\gamma_{\omega,\sigma}$ be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma_{\omega,\sigma}(t) = \left(\sqrt{\frac{(\omega^2-1)(\sigma^2+1)}{\omega^2\sigma^2}}, \frac{1}{\omega}\sqrt{\frac{\sigma^2+1}{\omega^2+\sigma^2}}\cosh\omega t, \frac{1}{\omega}\sqrt{\frac{\sigma^2+1}{\omega^2+\sigma^2}}\sinh\omega t,\right.$$
$$\left.\frac{1}{\sigma}\sqrt{\frac{\omega^2-1}{\omega^2+\sigma^2}}\sin\sigma t, \frac{1}{\sigma}\sqrt{\frac{\omega^2-1}{\omega^2+\sigma^2}}\cos\sigma t\right), \quad \sigma \neq 0.$$

Then $\gamma_{\omega,\sigma}$ is a helix with curvatures

$$k_1 = \sqrt{(\omega^2-1)(\sigma^2+1)}, \qquad k_2 = \frac{1}{2}\frac{\sigma^2-\omega^2+1}{\sqrt{(\omega^2-1)(\sigma^2+1)}}.$$

Example 5.9 (Helices of type 3). Let $\omega^2 > 1$ and let γ_ω be the curve in $\mathbb{H}_1^4$ defined by

$$\gamma_{\omega,\sigma}(t) = \left(\frac{\sqrt{\omega^4-1}}{\omega^2} + \frac{\sqrt{\omega^4-1}}{2(\omega^2+1)}t^2, \frac{1}{\omega^2}\cosh\omega t, \frac{1}{\omega^2}\sinh\omega t,\right.$$
$$\left.\sqrt{\frac{\omega^2-1}{\omega^2}}t, \frac{1-\omega^4}{2(\omega^2+1)}t^2\right).$$

Then γ_ω is a helix with curvatures

$$k_1 = \sqrt{\omega^2-1}, \qquad k_2 = -\tfrac{1}{2}\sqrt{\omega^2-1}.$$

Theorem 5.10 (Clasification theorem of 2-degenerate helices in $\mathbb{H}_1^4$). *Let γ be an s-degenerate Cartan curve fully immersed in $\mathbb{H}_1^4$. Then γ is a helix if and only if it is congruent to one in the families described in* Examples 5.7–5.9.

Proof. The idea of the proof is exactly alike as that in the precedent cases. Let $k_1 > 0$ and k_2 be the constant curvatures of γ. By the congruence theorem we only have to find a helix of one of the above types with these curvatures.

Case 1. Assume that $k_2 < -k_1/2$. Take the helix $\gamma_{\omega,\sigma}$ of type 1 determined by

$$\omega^2 = \tfrac{1}{2}((1-2k_1k_2) + \sqrt{(2k_1k_2+1)^2+4k_1^2}),$$
$$\sigma^2 = \tfrac{1}{2}((1-2k_1k_2) - \sqrt{(2k_1k_2+1)^2+4k_1^2}).$$

A straightforward computation shows that $0 < \sigma^2 < 1 < \omega^2$ and the curvatures of $\gamma_{\omega,\sigma}$ are k_1 and k_2.

Case 2. Assume that $k_2 > -k_1/2$. Take the helix $\gamma_{\omega,\sigma}$ of type 2 determined by

$$\omega^2 = \frac{1}{2}((1-2k_1k_2) + \sqrt{(2k_1k_2+1)^2 + 4k_1^2}),$$
$$\sigma^2 = \frac{1}{2}(-(1-2k_1k_2) + \sqrt{(2k_1k_2+1)^2 + 4k_1^2}).$$

As before we have that $\omega^2 > 1$ and the curvatures of $\gamma_{\omega,\sigma}$ are k_1 and k_2.

Case 3. Finally, assume that $k_2 = -k_1/2$. Take the helix γ_ω of type 3 determined by $\omega^2 = 1 + k_1^2$. It is easy to see that $\omega^2 > 1$ and the curvatures of γ_ω are k_1 and k_2. □

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